การกำกับแบบคี่บนด้านอย่างสวยงามของกราฟ PRISM(C_n) และ SHAFT(n, 1)

EDGE-ODD GRACEFUL LABELINGS OF PRISM(C_n) AND SHAFT(n, 1) GRAPHS

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บทคัดย่อ

กราฟ G ที่มี q ด้าน จะเรียกว่าเป็นกราฟที่มีการกำกับแบบคี่บนด้านอย่างสวยงาม ถ้ามีฟังก์ชัน f แบบสมนัยหนึ่งต่อหนึ่งจากด้านแต่ละด้านของกราฟไปยังเซต $\{1, 3, 5, ..., 2q-1\}$ โดยที่แต่ละจุดยอดจะกำกับด้วยผลรวมของจำนวนที่กำกับบนด้านทั้งหลายที่กระทบกับจุดยอดนั้น มอดุโล $2q$ และจำนวนที่เป็นผลรวมที่กำกับ แต่ละจุดยอดนั้นต่างกันทั้งหมด งานวิจัยชิ้นนี้แสดงว่า ปริซึมของวง C_n เมื่อ $n \geq 3$ เป็นกราฟที่มีการกำกับแบบคี่ บนด้านอย่างสวยงาม และกราฟที่เกิดจากการนำกราฟวงล้อ W_n สองวงมาเชื่อมจุดยอดกลางเข้าด้วยกัน ซึ่งจะเรียกว่ากราฟ Shaft(n, 1) เป็นกราฟที่มีการกำกับแบบคี่บนด้านอย่างสวยงาม เมื่อ n เป็นจำนวนเต็มคี่ที่มากกว่าหรือเท่ากับ 3

คำสำคัญ: การกำกับแบบคี่บนด้านอย่างสวยงาม, วง, ปริซึม, วงล้อ

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ABSTRACT

A graph G with q edges called an **edge-odd graceful graph** if there is a bijection f from the edges of the graph to the set {1, 3, 5, ..., $2q$–1} such that, when each vertex is assigned the sum of all the edges incident to it modulo $2q$, the resulting vertex labels are distinct. In this paper, we show that prism of cycle C_n, where $n \geq 3$, is an edge-odd graceful graph and two copies of wheel graphs W_n joining at the middle, which we call Shaft(n, 1), is an edge-odd graceful graphs whenever n is an odd integer and greater than or equal to 3.

Keywords: edge-odd graceful labeling, cycle, prism, wheel

Introduction

Let G be a simple undirected graph with q edges, In this article, we let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. In 1967, Rosa [3] gave a definition of a graceful labeling of G which is an injection f from $V(G)$ to the set {0, 1, 2, ..., q} such that each edge xy is assigned the label $|f(x)-f(y)|$, the resulting edge labels are distinct. In 1991, Gnanojothi [1] introduced an odd-graceful concept for a graph, that is an injection f from $V(G)$ to the set {0, 1, 2, ..., $2q$–1} such that, when each edge xy is assigned the label $|f(x)-f(y)|$, the resulting edges labels are in the set {1, 3, 5, ..., $2q$–1}. In 2009, Solairaju and Chithra [5] reversed the concepts of those two previous vertex labelings by defining an edge-odd graceful labeling as the following.

Definition 1.1 [4] Let G be a graph, with q edges an **edge-odd graceful labeling** of G is a bijection f from $E(G)$ to the set {1, 3, 5, ..., $2q$–1} so that the induced mapping f^* from $V(G)$ to the set {0, 1, 2, ..., $2q$–1} given by $f^*(x) = \sum f(xy) \pmod{2q}$ where the vertex x is adjacent to the vertex y. The edge labels and vertex labels are distinct. A graph that admitted an edge-odd graceful labeling is called **edge-odd graceful**.

Before we continue our exploration, let us introduce some of the common graphs that we will use in this article

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Before we continue our exploration, let us introduce some of the common graphs that we will use in this article

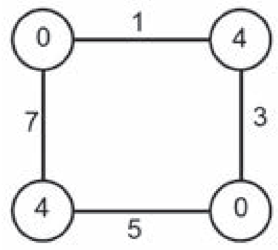
Definition 1.2 [2] A **path**, written P_n, is a simple graph whose n–vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. Then, the number of edges is $n - 1$.

Definition 1.3 [2] A **cycle**, written C_n, is a graph with an equal number of n –vertices and n –edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the cycle. In this thesis, we usually write a cycle C_n as $u_1u_2u_3...u_n$ and we named the vertices in the clockwise direction. Then, the number of edges is n.

Definition 1.4 [4] Let v_1, v_2, v_3, ..., v_n be vertices on the cycle of **SF(n, m)** and for each j $\in$ {1, 2, 3, ..., n}, the vertices v_j^1, v_j^2, v_j^3, ..., v_j^m be vertices joining v_j. That is, the vertex set of SF(n, m) is the set $\{v_j \mid j \in \{1, 2, 3, ..., n\}\} \cup \{v_j^i \mid j \in \{1, 2, 3, ..., n\}, i \in \{1, 2, 3, ..., m\}\}$ and the edge set is the set $\{v_j v_j^i \mid j \in \{1, 2, 3, ..., n\}, i \in \{1, 2, 3, ..., m\}\} \cup \{v_1v_n\}$. Then, the number of edges is $n + nm$.

Definition 1.5 [2] A **wheel graph** W_n is a graph with $n + 1$ vertices obtained by connecting a single vertex u to all vertices of a cycle $u_1u_2u_3...u_n$. Then, the vertex set of W_n is the set $\{u, u_1, u_2, u_3, ..., u_n\}$ and the edges set of W_n is the set $\{uu_i \mid i \in \{1, 2, 3, ..., n\}\} \cup \{u_{i+1}u_i \mid i \in \{1, 2, 3, ..., n-1\}\} \cup \{u_1u_n\}$. Then, the number of edges is $2n$.

According to Definition 1.1, one can see that some graphs are not edge–odd graceful. For example, C_4 is not an edge–odd graceful graph.



The above figure shows one of the labeling on C_4 which is not an edge–odd graceful labeling. Actually there are 6 possible ways to label the edge of C_4. One can easily check that the rest 5 permutations of {1, 3, 5, 7} on each edge will contain some repeated numbers labeling on the vertices.

Solairaju and Chithra [5] showed edge–odd graceful labeling of graphs related to paths. Later, Singhun [4] showed edge–odd graceful labeling of graphs related to cycles, $SF(n, m)$ where $n \geq 3$ and a wheel graph W_n where n is even. In Section 2 of this paper, we give algorithms for edge–labeling of graphs related to prism of cycle C_n, where $n \geq 3$, and two copies of wheel graphs W_n joining at the middle, which we later call Shaft(n, 1) for n is an odd integer and greater or equal to 3. In Section 3, we show that those algorithms give edge–odd graceful labeling of the Prism(C_n), where $n \geq 3$ and the Shaft (n, 1), where n is an odd integer and greater or equal to 3. In Section 4, we give a conclusion of our results and some discussion on the on–going research.

Methodology

Definition 2.1 Let $n \geq 3$ and C_n be an n–cycle $u_1u_2u_3...u_n$. Let $C'_n = u'_1u'_2u'_3...u'_n$ be a copy of C_n. Define **Prism(C_n)**, called the prism of C_n, by joining each corresponding vertices u_i of C_n to u'_i of C'_n. That is the edges of Prism(C_n) consists of $u_{i-1}u_i \in E(C_n)$, $u'_{i-1}u'_i \in E(C'_n)$ and $u_iu'_i$, bridges between C_n and C'_n. Thus, $E(\text{Prism}(C_n)) = E(C_n) \cup E(C'_n) \cup \{u_iu'_i \mid i \in \{1, 2, 3, ..., n\}\}$. Then, the number of edges is $q = n + n + n = 3n$.

Note that the Prism(C_n) can be viewed as a cartesian product of a cycle C_n and a path P_2.

Example 2.1 From Definition 2.1, we can have the Prism(C_4) as seen below.

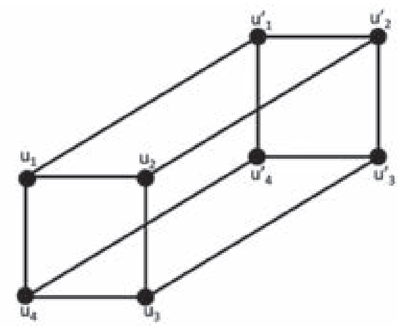


Figure 2.1 Prism(C_4)

The following diagram shows an edge labeling the Prism(C_3).

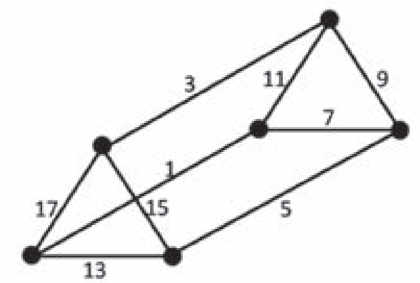


Figure 2.2 edge–labeling for Prism(C_3)

For any $n > 3$, we can label the edges of the Prism(C_n) by using the following algorithm.

Algorithm 2.1: For label each edge of Prism(C_n). Let G denote the Prism(C_n). Then,
$q = 3n$. Define f: $E(G) \longrightarrow \{1, 3, 5, ..., 6n–1\}$ by.

1.1 $f(u_{i-1}u_i) = 4n–2i + 1$, for $i \in \{2, 3, 4, ..., n\}$;

1.2 $f(u_1u_n) = 4n–1$;

1.3 $f(u'_{i-1}u'_i) = 6n - 2i + 1$, for $i \in \{2, 3, 4, ..., n\}$;

1.4 $f(u'_1u'_n) = 6n–1$;

1.5 $f(u_iu'_i) = 2i–1$, for $i \in \{1, 2, 3, ...,n\}$.

Example 2.2 From Algorithm 2.1, we can label each edge of the Prism(C_6) as shown below.

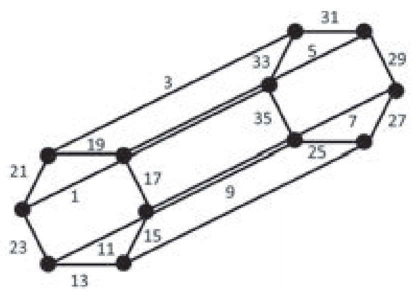


Figure 2.3 edge–labeling for Prism(C_6)

Definition 2.2 Let $n \geq 3$ and W_n be a wheel graph with $V(W_n) = \{u_1, u_2, u_3, \ldots, u_n, u\}$ and W'_n be a copy of W_n with the corresponding $V(W'_n) = \{u'_1, u'_2, u'_3, \ldots, u'_n, u'\}$. Define the **Shaft(*n*, 1)** by joining the vertices u of W_n to u' of W'_n. That is, $E(\text{Shaft}(n, 1)) = E(W_n) \cup E(W'_n) \cup \{uu'\}$. Then, the number of edges is $q = 2n + 2n + 1 = 4n + 1$.

Example 2.3 From Definition 2.2, we can have the Shaft(3,1) as seen below.

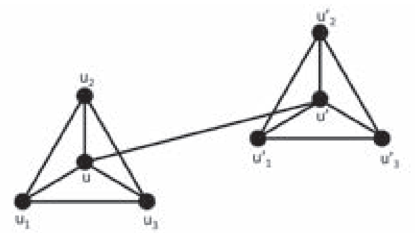


Figure 2.4 Shaft(3,1)

Next, we give an algorithm for labeling the edges of the Shaft(n, 1), where $n \geq 3$ is an odd integer.

Algorithm 2.2: For label each edge of Shaft(n, 1).Let $n \geq 3$ be an odd integer and G denote the Shaft(n, 1). Then,

$q = 4n + 1$. Define f: $E(G)$

$\longrightarrow \{1, 3, 5, \ldots, 8n + 1\}$ by

2.1 $f(uu') = 8n + 1$;

2.2 $f(u_1u_n) = 2n-1$;

2.3 $f(u_iu_{i+1}) = 2i-1$, for $i \in \{1, 2, 3, \ldots, n-1\}$;

2.4 $f(u_1u) = 2n + 1$;

2.5 $f(u_iu) = 4n-2i + 3$, for $i \in \{2, 3, 4, \ldots, n\}$;

2.6 $f(u'_1u'_n) = 6n-1$;

2.7 $f(u'_iu'_{i+1}) = 4n + 2i - 1$, for $i \in \{1, 2, 3, \ldots, n-1\}$;

2.8 $f(u'_1u') = 6n + 1$;

2.9 $f(u'_iu') = 8n - 2i + 3$, for $i \in \{2, 3, 4, \ldots, n\}$.

Example 2.4 From Algorithm 2.2, we can label each edge of the Shaft(5, 1) as shown below.

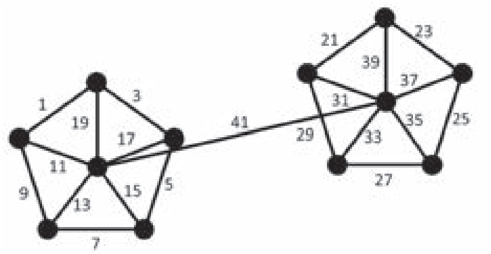


Figure 2.5 edge–labeling for Shaft(5, 1)

Results

Theorem 3.1 *The Prism(C_n) is an edge-odd graceful graph when $n \geq 3$.*

Proof. From Figure 2.2, we can see immediately that the induced vertex-labeling are shown below.

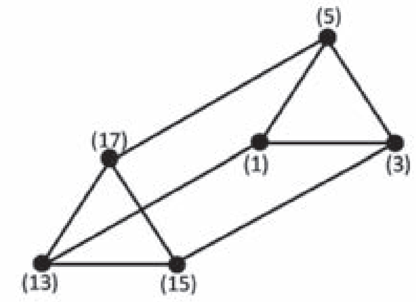


Figure 3.1 vertex-labeling induced from edge-labeling in Figure 2.2

Note that, we use integers with parenthesis to differentiate all vertex-labeling from the edge-labeling. Therefore, it is obvious from Figure 2.1 that the Prism(C_3) is an edge-odd graceful graph.

Let $n \geq 4$. We first prove that the function f defined in algorithm 2.1 is a bijection from $E(G)$ to $\{1, 3, 5, ..., 6n-1\}$. From algorithm 2.1(1.1, 1.2), we have

$A = \{f(u_{i-1}u_i), f(u_1u_n) \mid i \in \{2, 3, 4, ..., n\}\}$
$= \{2n + 1, 2n + 3, ..., 4n-1\}$.

From algorithm 2.1(1.3, 1.4), we have

$B = \{f(u'_{i-1}u'_i), f(u'_1u'_n) \mid i \in \{2, 3, 4, ..., n\}\}$
$= \{4n + 1, 4n + 3, ..., 6n-1\}$.

From algorithm 2.1(1.5), we have

$C = \{f(u_iu'_i) \mid i \in \{1, 2, 3, ..., n\}\} = \{1, 3, 5, ..., 2n-1\}$.

We can see clearly that A, B and C are disjoint and $f(E(\text{Prism}(C_n))) = A \cup B \cup C = \{1, 3, 5, ..., 6n-1\}$.

Next, we will show that the induced vertex-labeling from edge-labeling using algorithm 2.1 are in $\{0, 1, 2, ..., 6n-1\}$ and all distinct. From algorithm 2.1, we have

$$\begin{aligned}
f^+(u_1) &= f(u_1u'_1) + f(u_1u_n) + f(u_1u_2) \pmod{6n} \\
&= 1 + (4n - 1) + (4n - 3) \pmod{6n} \\
&= 2n-3. \\
f^+(u_n) &= f(u_nu'_n) + f(u_1u_n) + f(u_{n-1}u_n) \pmod{6n} \\
&= (2n-1) + (4n-1) + (2n + 1) \pmod{6n} \\
&= 2n-1. \\
f^+(u_i) &= f(u_iu'_i) + f(u_{i-1}u_i) + f(u_iu_{i+1}) \pmod{6n} \\
&= (2i - 1) + (4n-2i + 1) + \\
&\quad (4n-2(i + 1) + 1) \pmod{6n} \\
&= 2n - 2i - 1, \text{ for } i \in \{2, 3, 4, ..., n-1\}. \\
f^+(u'_1) &= f(u'_1u'_1) + f(u'_1u'_n) + f(u'_1u'_2) \\
&\quad \pmod{6n} = 1 + (6n - 1) + (6n - 3) \\
&\quad \pmod{6n} \\
&= 6n-3. \\
f^+(u'_n) &= f(u_nu'_n) + f(u'_1u'_n) + f(u'_{n-1}u'_n) \\
&\quad \pmod{6n} \\
&= (2n - 1) + (6n - 1) + (4n + 1) \\
&\quad \pmod{6n} \\
&= 6n-1. \\
f^+(u'_i) &= f(u_iu'_i) + f(u'_{i-1}u'_i) + f(u'_iu'_{i+1}) \pmod{6n} \\
&= (2i - 1) + (6n-2i + 1) + \\
&\quad (6n-2(i + 1) + 1) \pmod{6n} \\
&= 6n - 2i - 1 \pmod{6n}, \\
&\quad \text{for } i \in \{2, 3, 4, ..., n-1\}.
\end{aligned}$$

We can see that $\{f^*(u_i) \mid i \in \{1, 2, 3, ..., n\}\} = \{2n-3\} \cup \{1, 3, 5, ..., 2n-5\} \cup \{2n-1\}$ and $\{f^*(u'_i) \mid i \in \{1, 2, 3, ..., n\}\} = \{4n+1, 4n+3, 4n+5, ..., 6n-5, 6n-3, 6n-1\}$. It is clear that if $n \geq 4$, these two sets are all distinct and both are subsets of $\{0, 1, 2, ..., 6n-1\}$. Therefore, the function f defined in algorithm 2.1 is an edge–odd graceful labeling and the Prism(C_n) is an edge–odd graceful graph for all $n \geq 3$.

Example 3.1 From the edge–labeling in Example 2.2, the induced vertex–labeling of the Prism(C_6) is shown below.

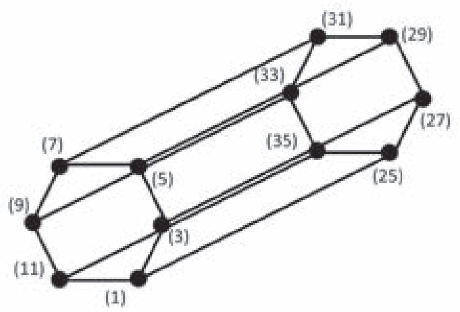


Figure 3.2 vertex–labeling induced from edge–labeling in example 2.2

Theorem 3.2 *The Shaft(n, 1) is an edge–odd graceful graph when n is odd and* $n \geq 3$.

Proof. To prove that f in algorithm 2.2 is a bijection from $E(G)$ to $\{1, 3, 5, ..., 8n+1\}$, we consider the followings. From algorithm 2.2(2.1), we have

$A = \{f(uu')\} = \{8n+1\}$.

From algorithm 2.2(2.2, 2.3), we have

$B = \{f(u_iu_{i+1}), f(u_1u_n) \mid i \in \{1, 2, 3, ..., n-1\}\}$
$= \{1, 3, 5, ..., 2n-1\}$.

From algorithm 2.2(2.4, 2.5), we have

$C = \{f(u_1u), f(u_iu) \mid i \in \{2, 3, 4, ..., n\}\}$
$= \{2n+1, 2n+3, 2n+5, ..., 4n-1\}$.

From algorithm 2.2(2.6, 2.7), we have

$D = \{f(u'_iu'_{i+1}), f(u'_1u'_n) \mid i \in \{1, 2, 3, ..., n-1\}\}$
$= \{4n+1, 4n+3, 4n+5, ..., 6n-1\}$.

From algorithm 2.2(2.8, 2.9), we have

$E = \{f(u'_iu'), f(u'_1u') \mid i \in \{2, 3, 4, ..., n\}\}$
$= \{6n+1, 6n+3, 6n+5, ..., 8n-1\}$.

We can see clearly that A, B, C, D and E are disjoint and $f(E(\text{Shaft}(n, 1))) = A \cup B \cup C \cup D \cup E = \{1, 3, 5, ..., 8n+1\}$. Next, we will show that the induced vertex–labeling from edge–labeling using algorithm 2.2 are in $\{0, 1, 2, ..., 8n+1\}$ and all distinct. From algorithm 2.2, we have

$f^*(u) = 3n^2 - 1 \pmod{8n+2}$.
$f^*(u') = 7n^2 - 1 \pmod{8n+2}$.
$f^*(u_i) = 4n + 2i - 1$, for $i \in \{1, 2, 3, ..., n\}$.
$f^*(u'_1) = 8n-1$.
$f^*(u'_2) = 8n+1$.
$f^*(u'_i) = 2i-5$, for $i \in \{3, 4, 5, ..., n\}$.

We can see that $\{f^*(u_i) \mid i \in \{1, 2, 3, ..., n\}\} = \{4n+1, 4n+3, 4n+5, ..., 6n-1\}$ and $\{f^*(u'_i) \mid i \in \{1, 2, 3, 4, ..., n\}\} = \{8n-1\} \cup \{8n+1\} \cup \{1, 3, 5, ..., 2n-5\}$. It is clear that if n is an odd integer and $n \geq 3$, these two sets are all distinct and both are subsets of $\{0, 1, 2, ..., 8n+1\}$. We can see that the vertex–labeling of vertices $\{u_1, u_2, u_3, ..., u_n\}$ and $\{u'_1, u'_2, u'_3, ..., u'_n\}$ are odd integers. However, the vertex–labeling of vertices $\{u, u'\}$ are even integers. Finally, we need to show that $f^*(u)$ and $f^*(u')$ are distinct under modulo $8n + 2$. Suppose in a contrary that $f^*(u) \equiv f^*(u')$ (mod $8n + 2$).

Then, $3n^2 - 1 \equiv 7n^2 - 1$ (mod $8n + 2$). That is, $2n^2 \equiv 0$ (mod $4n + 1$). There exists an integer k such that $2n^2 = (4n + 1)k$. Since $2n^2$ is even and $4n + 1$ is odd, k is even, say $k = 2l$ for some integer l. The quadratic formula implies that $16l^2 + 4l = 4l(4l + 1)$ must be square. This is a contradiction since gcd $(4l, 4l + 1) = 1$. Hence, $f^*(u)$ and $f^*(u')$ are distinct.

Therefore, the function f defined in algorithm 2.2 is an edge-odd graceful labeling and the Shaft(n, 1) is an edge-odd graceful graph for n is odd and $n \geq 3$.

Example 3.2 From the edge-labeling in Example 2.4, the induced vertex-labeling of the Shaft(5, 1) is shown below.

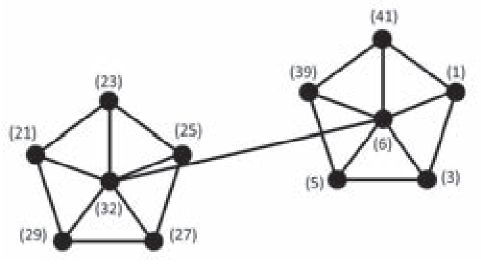


Figure 3.3 vertex-labeling induced from edge-labeling in example 2.4

Conclusion and Discussion

For the Prism(C_n), where $n = 3$, we can use the edge-labeling shown in Figure 2.2 to show that the Prism(C_3) is an edge-odd graceful graph. For $n \geq 4$, the edge-labeling given in Algorithm 2.1 can be proved that the Prism(C_n) is an edge-odd graceful graph. Finally, as we define a new Shaft(n, 1), for $n \geq 3$, we can show that if n is an odd integer and $n \geq 3$, the Shaft(n, 1) is an edge-odd graceful graph using the edge-labeling given by Algorithm 2.2 One may extend the investigation by trying to find an algorithm for edge-labeling that makes the Shaft(n, 1) to be an edge-odd graceful graph for even integer n with $n \geq 4$. The followings figures show edge labelings for Shaft(4, 1) and Shaft(6, 1) in which we can see that they are edge-odd graceful graphs. However, we still cannot find the general algorithm for them.

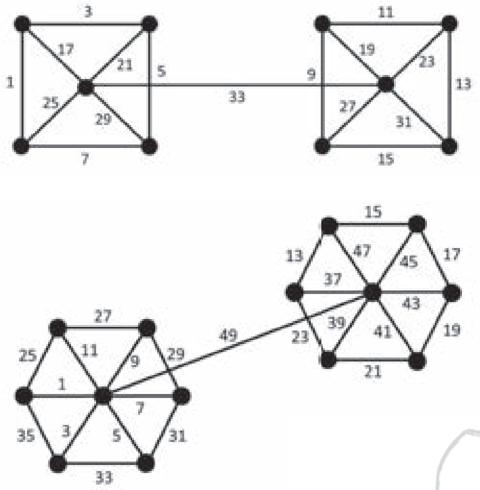


Figure 3.4 Edge-labeling for Shaft(4, 1) and Shaft(6, 1)

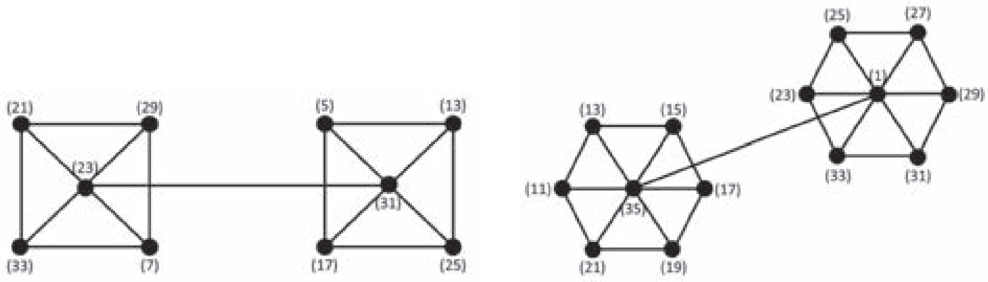


Figure 3.5 Vertex-labeling induced from edge-labeling in Figure 3.4

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